Wegner estimate and the density of states of some indefinite alloy type Schrödinger Operators

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Abstract

We study Schrödinger operators with a random potential of alloy type. The single site potentials are allowed to change sign. For a certain class of them we prove a Wegner estimate. This is a key ingredient in an existence proof of pure point spectrum of the considered random Schrödinger operators. Our estimate is valid for all bounded energy intervals and all space dimensions and implies the existence of the density of states.

Keywords: density of states, random Schrödinger operators, Wegner estimate, multi scale analysis, localization, indefinite single site potential

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1 Alloy type models and Wegner's estimate

The subject matter of this work are families of random Schrödinger operators $\{H_\omega\}_{\omega\in\Omega}$ acting on $L^2(\mathbb{R}^d)$. They have been introduced as quantum mechanical models for disordered media in solid state physics. The random Schrödinger operator we consider is of *Anderson* or *alloy* type

$$H_\omega = -\Delta + V_0 + V_\omega, \tag{1}$$

where the negative Laplace operator $-\Delta$ corresponds to the kinetic energy, V_0 is a bounded $\mathbb{Z}^d$-periodic potential and V_ω is the random potential given

by the stochastic process

$$V_\omega(x) = \sum_{k\in\mathbb{Z}^d} \omega_k\, u(x-k)\ . \tag{2}$$

The function $u\colon \mathbb{R}^d \to \mathbb{R}$ is called *single site potential* and represents the contribution to V_ω due to a single nucleus or ion situated at a lattice point $k \in \mathbb{Z}^d$. We assume that $u \in L^p(\mathbb{R}^d)$ with $p = 2$ for $d \le 3$ and $p > d/2$ for $d \ge 4$ is compactly supported. The ω_k are real-valued, random *coupling constants*. I.e. while we fix the shape of the single site potential at each $k \in \mathbb{Z}^d$, its strength is allowed to vary randomly. The random variables $\omega_k, k \in \mathbb{Z}^d$ are independent and identically distributed and the distribution measure μ of ω_0 has a density f. We consider the coupling constants as components of a random vector $\omega := \{\omega_k\}_{k\in\mathbb{Z}^d} \in \Omega := \times_{k\in\mathbb{Z}^d}\mathbb{R}$. The probability space Ω is equipped with the product measure $\mathbb{P} := \bigotimes_{k\in\mathbb{Z}^d} \mu$. The corresponding expectation is denoted by $\mathbb{E}$.

To state our main technical result we introduce auxiliary objects associated to finite cubes. We denote by Λ_l the cube of side length l and centre at 0 and with H^l_ω the restriction of H_ω to Λ_l with periodic boundary conditions (b.c.). Note that H^l_ω has purely discrete spectrum so we can enumerate its eigenvalues $\lambda_i(H^l_\omega)$ in non-decreasing order and counting multiplicities. For a bounded interval $I \subset \mathbb{R}$ the spectral projection $P^l_\omega(I)$ of H^l_ω has a finite trace.

Theorem 1 (Wegner estimate) *Let the density f have compact support and be in the Sobolev space $W^1_1(\mathbb{R})$ and the single site potential be of* generalized step function *form:*

$$u(x) = \sum_{k\in\Gamma} \alpha_k\ w(x-k),\ \ \Gamma \subset \mathbb{Z}^d, \tag{3}$$

where $w \ge \kappa\chi_{[0,1]^d}$ with some positive κ and $w \in L^p(\mathbb{R}^d)$ with $p = 2$ for $d \le 3$ and $p > d/2$ for $d \ge 4$. We assume that Γ is a finite set and

$$\alpha^* = \sum_{k\neq 0} |\alpha_k| < |\alpha_0|. \tag{4}$$

Then we have for all $E \in \mathbb{R}$

$$\mathbb{E}\left[\operatorname{Tr} P^l_\omega([E-\epsilon, E])\right] \le const\ \epsilon\ l^d, \quad \forall\, \epsilon \ge 0. \tag{5}$$

The constant depends on E but not on ϵ.

The theorem remains true if we replace the periodic b.c. by Dirichlet or Neumann ones. We call α the *convolution vector*.

Remark 2 For our proof it is essential that the single site potential u is of generalized step function form, since this enables us to work simultaneously with two different representations of the random potential, cf. Section 4. Condition (4) ensures the invertibility of the block-Toeplitz operator generated by the convolution vector α and moreover a uniform bound on the norms of the inverses of finite truncations of this Toeplitz operator. This uniform invertibility could be alternatively ensured by an appropriate condition on the symbol of the Toeplitz operator, cf. e.g. Chapter 7 of [3] or Chapters 2 and 6 of [4]. This will be discussed elsewhere, as announced in [30]. Recently there has been increased interest in conditions on the symbol of the Toeplitz operator, which ensure merely that the norms of the inverses of finite size truncations grow at most polynomially in the size of the truncation, cf. [40, 38, 37, 39, 2]. Such conditions could be useful for the derivation of a Wegner estimate which, in turn, can be used as an ingredient od a proof of localization, although it is not sufficient to ensure the existence of the density of states, cf. Section 2.

In the next section we deduce the existence of the density of states from the Wegner estimate in Theorem 1 and discuss its role for the proof of localization. Furthermore we review earlier results for indefinite alloy type models. Sections 3 to 5 contain the proof of the main technical Theorem 1 and the last two sections are devoted to the discussion of generalizations of the results and the application to localization for indefinite models.

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2 Density of states and localization

Under our assumptions the family $H_\omega, \omega \in \Omega$ fits into the general theory of ergodic random Schrödinger operators [22, 6, 35]. We infer two central

results from this theory.

(A) The spectrum of the family $H_\omega, \omega \in \Omega$ is non-random in the following sense. There exists a subset Σ of the real line and an $\Omega' \subset \Omega$, $\mathbb{P}(\Omega') = 1$ such that for all $\omega \in \Omega'$ one has $\sigma(H_\omega) = \Sigma$. The analogous statement holds true for the essential, discrete, continuous, absolutely continuous, singular continuous, and pure point part of the spectrum. Note that the pure point spectrum σ_{pp} is the closure of the set of eigenvalues of H_ω.

(B) There exists a *self averaging* integrated density of states associated with the family $H_\omega, \omega \in \Omega$. This means that the normalized eigenvalue counting functions

$$N_\omega^l(E) = l^{-d} \#\{i |\ \lambda_i(H_\omega^l) < E\} = l^{-d} \operatorname{Tr} P_\omega^l(]-\infty, E[) \tag{6}$$

of H_ω^l converge for almost all ω to a limit $N := \lim_{l\to\infty} N_\omega^l$ which is ω-independent.

We call N the *integrated density of states* (IDS) of H_ω and N_ω^l the *finite volume* IDS of H_ω^l.

Remark 3 While the two above facts (A) and (B) follow from the general theory, one is interested in more detailed spectral properties of specific models $H_\omega, \omega \in \Omega$, e.g.:

- Which spectral types can occur in $\sigma(H_\omega)$?
- Can something be said about the regularity of the IDS N as a function of the energy E? Is it Hölder continuous or does even its derivative, the *density of states* exist.

Our result on the regularity of the IDS is strong enough to imply the existence of the density of states:

Theorem 4 (Density of states) *Under the assumptions of Theorem 1 the IDS of the alloy type model $\{H_\omega\}_{\omega\in\Omega}$ is Lipschitz continuous: for all $E \in \mathbb{R}$ there exists a constant C such that*

$$N(E) - N(E - \epsilon) \le C\,\epsilon, \quad \forall\ \epsilon > 0\ . \tag{7}$$

By Rademacher's theorem it follows that the derivative $\frac{dN}{dE}$ exists for almost all E.

The analog result for $u \geq \kappa\chi_{[0,1]^d}, \kappa > 0$ is proved in [8], cf. also Section 5.

Remark 5 The theorem follows directly from (5) and the self averaging property $N(\cdot) = \mathbb{E}N(\cdot)$. There is a explicit upper bound for the density of states, see (28).

The second question of Remark 3 is related to the transport properties of the medium modelled by H_ω. A perfect crystal is described by a Schrödinger operator with periodic potential. It has purely absolutely continuous spectrum, which reflects its good electric transport properties. In contrast to this, it has been proven that random perturbations of this regular structure give rise to energy intervals with pure point spectrum. This corresponds to the less effective transport properties of random media. The existence of pure point spectrum in this context is called *(Anderson) localization*.

Now we indicate the general scheme of the proof of localization and where the Wegner estimate enters. In section 7 we show that Theorem 1 implies localization for some alloy type models with single site potentials that change sign.

A powerful tool for proving localization is the so-called *multi scale analysis* (MSA), an induction argument over increasing length scales $l_k, k \in \mathbb{N}$. This technique was first applied by Fröhlich and Spencer [19] to the discretization of the Schrödinger operator (1) and underwent since then a number of strengthenings [34, 18], simplifications [47] and adaptations to the continuous model on $L^2(\mathbb{R}^d)$ [33, 27, 8], which we are considering. More recently it was used also for Hamiltonians governing the motion of classical waves [14, 15, 41, 12].

At the same time extensive research has been done to identify physical situations where one can prove the key ingredients needed to start and carry trough the MSA [28, 1, 26, 25, 44, 17].

Remark 6 (MSA Hypotheses) Let us fix some notation. For points $x \in \mathbb{R}^d$ in the configuration space let $\|x\|_\infty := \sup\{|x_i|, i = 1, \dots, d\}$ denote the sup-norm. Let $\delta > 0$ be a small constant independent of the length scale l_k and $\phi_k(x) \in C^2$ a function which is identically equal to 0 for x with $\|x\|_\infty > l_k - \delta$ and identically equal to one for x with $\|x\|_\infty < l_k - 2\delta$. The commutator $W(\phi_k) := [-\Delta, \phi_k] := -(\Delta\phi_k) - 2(\nabla\phi_k)\nabla$ is a local operator acting on functions which live on a ring of width δ near the boundary of $\Lambda_k := \Lambda_{l_k}$. We say that a pair $(\omega, \Lambda_k) \in \Omega \times \mathcal{B}(\mathbb{R}^d)$ is *m-regular* for a given energy E, if

$$\sup_{\epsilon \neq 0} \|W(\phi_k)(H_\omega^l - E + \epsilon i)^{-1}\chi^{l_k/3}\|_{\mathcal{L}(L^2)} \leq e^{-ml_k} \ . \tag{8}$$

Here $\chi^{l_k/3}$ is the characteristic function of $\Lambda_{l_k/3} := \{y | \, \|y\|_\infty \leq l_k/6\}$. Thus the distance of the supports of $\nabla\phi_k$ and $\chi^{l_k/3}$ is at least $l_k/3 - 2\delta \geq l_k/4$.

There are two key hypotheses for the MSA associated to energies E in the interval $I \subset \mathbb{R}$ in which one wants to prove the existence of pure point spectrum.

(H1) $\Leftrightarrow$ There exist constants $Q_1 \in]0, \infty[, m \in]Q_1^{-1}, \infty[, q > 0$ such that

$$\mathbb{P}\{\omega | \, (\omega, \Lambda_{Q_1}) \text{ is } m\text{-regular}\} \geq 1 - Q_1^q. \tag{9}$$

(H2) $\Leftrightarrow$ There exist constants $Q_2, \eta_0, \in]0, \infty[$ such that

$$\mathbb{P}\{\omega | \, d(\sigma(H_\omega|_\Lambda), E) \leq \eta\} \leq C_W \, \eta \, |\Lambda| \tag{10}$$

for all boxes Λ with side length larger than Q_2 and all $\eta \leq \eta_0$.

The first hypothesis (H1) is commonly called *initial scale estimate*. It provides the induction anchor for the MSA. Most papers deduce (H1) from the asymptotic behaviour of the IDS at so-called spectral *fluctuation boundaries*. This asymptotics reflect the fact that "electron levels" are very sparse near such edges of the spectrum. The existence of these tails has been first deduced on physical grounds by Lifshitz [32].

The estimate (H2) is associated with a paper of Wegner [49] where he — like Fröhlich and Spencer [19] — considers Schrödinger operators on $l^2(\mathbb{Z}^d)$. Wegner's estimate is needed to draw the induction conclusion on each length scale $l_k, k \in \mathbb{N}$ of the MSA. This is the reason why — in contrast to (H1) — it has to be valid for arbitrarily large scales $l \geq Q_2$ [1].

Once the MSA has been accomplished, one proceeds to prove localization using the spectral averaging technique and expansion in generalized eigenfunctions, cf. [8] or Sections 7 and 8 in [26]. For a different version of the MSA see [42].

Actually for the MSA a variety of weaker bounds than (10) is sufficient. It is enough to know

(H2') $\Leftrightarrow$ There exist constants $Q_2, \eta_0, a, b \in]0, \infty[$ such that

$$\mathbb{P}\{\omega | \, d(\sigma(H_\omega|_\Lambda), E) \leq \eta\} \leq C_W \, \eta^a |\Lambda|^b \tag{11}$$

for all boxes Λ with side length larger than Q_2 and all $\eta \leq \eta_0$. Inequality (11) is implied by the Hölder continuity of the averaged finite volume IDS,

[1] Each hypothesis in Remark 6 has its own initial scale: Q_1 and Q_2. On the other hand the MSA itself needs a sufficiently large starting scale Q_0. For the whole argument to work out one has to make sure that Q_1 can be chosen at least as large as the maximum of Q_0 and Q_2.

mentioned in Remark 3:

$$\mathbb{E}\{N^l_\bullet(E+\eta) - N^l_\bullet(E-\eta)\} \le C_W \, \eta^a |\Lambda_l|^{b-1} \ . \tag{12}$$

We discuss briefly related results on Wegner estimates and localization for single site potentials with changing sign.

In [28] Klopp proves a Wegner estimate like (H2') for the alloy-type model (1) at low energies. It is valid for energy intervals $[E-\eta, E+\eta] \subset [-\infty, E_c]$ where E_c is an energy strictly above the infimum of the spectrum of H_ω. The result applies to arbitrary dimensions d, and for the single site potential u only some mild regularity and decay assumptions are required, but no sign-definiteness. The density f has to belong to a nice class of functions which contains as a subset $C^1(\mathbb{R})$. The paper [20] of Hislop and Klopp improves the volume dependence of the Wegner estimate in [28] using results from [11] and extends the validity of the estimate to energy values near internal spectral edges.

The techniques developed in [43, 5] by Stolz, resp. Buschmann and Stolz for one-dimensional Schrödinger operators allow to deduce localization at all energies without proving a Wegner estimate. The method applies to potentials of Poissonian or random displacement type as well as to the alloy-type model (1) in one dimension with no sign restrictions on the single site potential u, cf. also [13].

3 From the finite volume IDS to localized spectral projections

In this section we reduce the bound of the finite volume IDS to averaging of spectral projections localized in space. We follow the arguments of [8, Section 4] which in turn is a generalization of [31]. Let $I :=]E_1, E_2[$ be an open energy interval, $P^l_\omega(I)$ the spectral projection of H^l_ω onto the interval I and let $\mathrm{Tr}(A)$ denote the trace of an operator A. Without loss of generality we assume $w = \kappa \chi_{[0,1]^d}$ since only the lower bound matters. Moreover, by rescaling the density f we can achieve $\kappa = 1$. For the finite volume IDS and any $\epsilon > 0$ we have

$$\mathbb{E}\left[N^l_\omega(E_2) - N^l_\omega(E_1+\epsilon)\right] \le \frac{1}{l^d}\mathbb{E}\left[\mathrm{Tr}\, P^l_\omega(I)\right] \ . \tag{13}$$

Let $\tilde\Lambda := \Lambda \cap \mathbb{Z}^d$ be the lattice points in Λ. As in [8] we estimate

$$\mathbb{E}\left[\mathrm{Tr}\, P^l_\omega(I)\right] \le e^{E_2} C_V \sum_{j\in\tilde\Lambda} \left\|\mathbb{E}\left[\chi_j P^l_\omega(I)\chi_j\right]\right\| \tag{14}$$

where χ_j is the characteristic function of the unit cube centered at j and the constant C_V is an uniform upper bound on $\mathrm{Tr}(\chi_j e^{-H_\omega^{\Lambda+j}}\chi_j)$, cf. proof of Theorem 76 in [36]. Here $\Lambda+j$ denotes the unit cube centered at $j \in \mathbb{Z}^d$ and $H_\omega^{\Lambda+j}$ the restriction of H_ω on this cube with Neuman b.c. For the bound on the operator norm in (14) it is sufficient to consider $\mathbb{E}[\langle \phi, \chi_j P_\omega^l(I)\chi_j\phi\rangle]$ for all normalized $\phi \in L^2(\Lambda_l)$.

4 Transformation of variables

In this section we introduce a transformation of variables on the probability space Ω. It will enable us to use a spectral averaging result from [8] to bound the expectation value on the rhs of (14). At the same time we have to keep control of the new probability density, which will lose its simple product structure by the transformation.

Let $A := \{a_{j,k}\}_{j,k\in\mathbb{Z}^d}$ be an infinite Toeplitz matrix with entries $a_{j,k} = \alpha_{j-k}$. It transforms the components of the random vector linearly: $\eta := A\omega \in \Omega$. Note that due to the assumptions on the vector α the matrix A is invertible and one can derive a bound on its inverse by a Neumann series. The row-sum matrix norm we use is given by $\|A\| := \|A\|_1 := \sup_{j\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}|a_{j,k}|$. Since A is a Toeplitz matrix generated by the vector α we can write it as

$$A =: \mathrm{Id} + S, \ \ \|S\| = \sum_{j\neq 0}|\alpha_j| = \alpha^* < |\alpha_0| \tag{15}$$

by the definition of α^*. By rescaling we can assume $\alpha_0 = 1$. So A^{-1} exists and we have $\|A^{-1}\| \leq \frac{1}{1-\alpha^*}$. Now we introduce truncations A_Λ of the matrix A associated to a cube $\Lambda \subset \mathbb{R}^d$. Denote with $\Lambda^+ = \tilde{\Lambda} - \Gamma = \{\lambda - \gamma | \ \lambda \in \tilde{\Lambda}, \gamma \in \Gamma\}$ the set of all sites k whose associated potentials $u(\cdot - k)$ influence the potential V_ω within the cube Λ. The truncated matrix $A_\Lambda = \{\alpha_{j-k}\}_{j,k\in\Lambda^+}$ acts on $\omega_\Lambda = \{\omega_k, k \in \Lambda^+\}$ to give a vector $\eta_\Lambda = \{\eta_k, k \in \Lambda^+\}$

$$(\eta_\Lambda)_j = (A_\Lambda\omega_\Lambda)_j = \sum_{k\in\Lambda^+}\alpha_{j-k}(\omega_\Lambda)_k \ . \tag{16}$$

The decomposition and the bound in (15) remain true for the truncations A_Λ.

We write now the restricted Schrödinger operator H_ω^l in the new variables η. We drop for the remainder of this section the index in the truncated matrix A_Λ and vectors $\omega_\Lambda, \eta_\Lambda$, and write $\tilde{V}_\eta$ for $V_\omega = V_{A^{-1}\eta}$, $\tilde{H}_\eta^l$ for $H_\omega^l =$

$H^l_{A^{-1}\eta}$ and similarly $\tilde{P}^l_\eta = P^l_\omega$ for the operators living on $\Lambda = \Lambda_l$. For $x \in \Lambda$ we have

$$
\begin{aligned}
V_\omega(x) &= \sum_{k\in\Lambda^+} \omega_k \sum_{l\in\Gamma} \alpha_l\, \chi_{k+l}(x) = \sum_{j\in\tilde{\Lambda}} \chi_j(x) \sum_{k\in\Lambda^+} \alpha_{j-k}\omega_k \\
&= \sum_{j\in\tilde{\Lambda}} \eta_j \chi_j(x) = \tilde{V}_\eta(x)\ . \qquad (17)
\end{aligned}
$$

The common density of the random variable $\eta_k, k \in \Lambda^+$ is given by

$$
k(\eta) = |\det A^{-1}|\, F(A^{-1}\eta),\ \ F(\omega) = \prod_{k\in\Lambda^+} f(\omega_k) \qquad (18)
$$

5 Bounds on the density

The structure of the random operator $\tilde{H}^l_\eta$ in the new variables makes it easier to estimate the expectation value

$$
\mathbb{E}\left[\langle\phi, \chi_j P^l_\omega(I)\chi_j\phi\rangle\right] = \mathbb{E}\left[\langle\phi, \chi_j \tilde{P}^l_\eta(I)\chi_j\phi\rangle\right]. \qquad (19)
$$

We single out the lattice point $j \in \tilde{\Lambda}$ and consider the one-parameter family of operators

$$
\eta_j \mapsto \tilde{H}^l_{\hat{\eta}}(\eta_j)\ , \qquad (20)
$$

where $\hat{\eta} = \{\eta_k, k \in \Lambda^+ \setminus \{j\}\}$ and $\tilde{H}^l_{\hat{\eta}}(\eta_j) = \tilde{H}^l_\eta$. Similarly we write $k_{\hat{\eta}}(\eta_j) = k(\eta)$ for the common density. Locally on the cube $\Lambda_1 + j$ the dependence (20) on the parameter η_j is strictly increasing. The price we had to pay is that the random variable η_j is (negatively) correlated to components of $\hat{\eta}$. Set $L = \#\Lambda^+$. For a normalized vector $\phi \in L^2(\Lambda_l)$ set

$$
s(\eta) := \langle\phi, \chi_j \tilde{P}^l_\eta(I)\chi_j\phi\rangle\ . \qquad (21)
$$

By Fubini's theorem we have for (19)

$$
\int_{\mathbb{R}^L} d\eta\, k(\eta)\, s(\eta) = \int_{\mathbb{R}^{L-1}} d\hat{\eta} \int_{\mathbb{R}} d\eta_j\, k(\eta)\, s(\eta) \qquad (22)
$$

We bound the rhs of (22) as in [8, Section 4].

$$
\int_{\mathbb{R}} d\eta_j\, k(\eta)\, s(\eta) \le |I|\ \|k_{\hat{\eta}}(\cdot)\|_\infty. \qquad (23)
$$

The bound is valid for f without compact support, too, as pointed out in [16]. Furthermore the boundedness condition on w in [8] can be replaced with relative boundedness, which is ensured by $w \in L^p$ with appropriate p. Note that we have normalized the random potential V_ω in such a way that the constant c_0 from [8] is equal to 1.

Using Fubini's theorem in the reverse direction and transforming back to the ω-variables we estimate (19) by

$$\begin{aligned} |I| \int_{\mathbb{R}^{L-1}} d\hat{\eta}\, \|k_{\hat{\eta}}(\cdot)\|_\infty &\le |I| \int_{\mathbb{R}^{L-1}} d\hat{\eta} \int_{\mathbb{R}} d\eta_j\ |k'_{\hat{\eta}}(\eta_j)| \\ &\le |I| \int_{\mathbb{R}^L} d\eta\ |k'_{\hat{\eta}}(\eta_j)| = |I|\,|\det A| \int_{\mathbb{R}^L} d\omega\, |k'_{\hat{A\omega}}\,[(A\omega)_j]\,|. \end{aligned}$$

Let $B = \{b_{i,j}\}_{i,j\in\Lambda^+}$ denote the inverse of $A = A_\Lambda$. We calculate the derivative of k with respect to η_j

$$k'_{\hat{A\omega}}[(A\omega)_j] = |\det B| \sum_{k\in\Lambda^+} f'(\omega_k)\ b_{k,j} \prod_{\substack{i\in\Lambda^+\\ i\neq k}} f(\omega_i) \tag{24}$$

and the corresponding integral

$$\int_{\mathbb{R}} d\omega\ |k'_{\hat{A\omega}}[(A\omega)_j]| \quad \le \quad |\det B|\, \|f'\|_{L^1} \sum_{k\in\Lambda^+} |b_{k,j}|. \tag{25}$$

This gives for the expectation (19) the estimate

$$\mathbb{E}\left[\langle\phi, \chi_j P^l_\omega(I)\chi_j\phi\rangle\right] \le |I|\ \|f'\|_{L^1} \sum_{k\in\Lambda^+} |b_{k,j}|. \tag{26}$$

By the bound (15) on the inverse of A we know $\sum_{k\in\Lambda^+} |b_{k,j}| \le \|B\| \le (1-\alpha^*)^{-1}$. So (19) is bounded by $|I|\,\|f'\|_{L^1}(1-\alpha^*)^{-1}$ which is independent of Λ_l and j. The average trace in (14) is thus bounded by

$$e^{E_2} C_V (1-\alpha^*)^{-1}\, \|f'\|_{L^1}\, |I|\, |\Lambda|\ . \tag{27}$$

Thus we proved that the averaged finite volume IDS is Lipschitz continuous

$$\mathbb{E}\left[N^l_\bullet(E_2) - N^l_\bullet(E_1+\epsilon)\right] \le C\,|E_2 - E_1|,\ \forall\ \epsilon > 0 \tag{28}$$

with $C := e^{E_2} C_V \frac{1}{1-\alpha^*}\, \|f'\|_{L^1}$. By the Čebyšev inequality estimate (H2) now follows.

6 Generalizations

We consider some generalizations of Theorem 1. Details can be found in [46].

Remark 7 (Discrete model) One could also consider the discrete Schrödinger operator

$$h_\omega = -\Delta_{\rm disc} + V_\omega \text{ on } l^2(\mathbb{Z}^d) \tag{29}$$

where $[-\Delta_{\rm disc}\phi](i) = \sum_{|i-n|=1}(\phi(i) - \phi(n))$ and in the definition of the multiplication operator V_ω the characteristic function of the unit cube $\chi_0(x)$ is replaced by the Kronecker symbol $\delta_0(i)$.

Our proof works for this model, too, since the results in [8, Section 4] are formulated for abstract one-parameter families of operators. In Sections 3 to 5 of this paper we would just have to change the notations.

Remark 8 (Correlated potentials) We can regard Theorems 1 as a result about alloy type Schrödinger operators with non-negative single site potential u but negatively correlated coupling constants. I.e. we consider $\tilde{H}_\eta$ as the original operator.

Wegner estimates for dependent coupling constants with bounded conditional densities were derived in [10] (cf. also [21]).

Correlated random potentials are also treated in the papers [48, 25] on the discrete Anderson model and the alloy type model. There the long range correlations are studied, and the way the MSA has to be adapted to yield localization in this case.

Remark 9 (More general convolution vectors α) The condition $\alpha^* < |\alpha_0|$ in Theorem 1 can be relaxed as can be seen from the following example.

Example 10 Let $e = (1, 0, \ldots, 0) \in \mathbb{Z}^d$ and $u = \chi_0 - \chi_e$ be the single site potential of H_ω (1). This corresponds to $\alpha_0 = 1, \alpha_e = -1$ and $\alpha_k = 0$ otherwise. The truncations A_Λ have inverses $B_\Lambda := \{b_{j,k}\}_{j,k\in\Lambda^+}$ with entries $b_{j,k} = 1$ for $j, k \in \Lambda^+$, $k_1 \le j_1$ and $k_i = j_i$ for $i = 2, \ldots, d$ and $b_{j,k} = 0$ otherwise. Here for a $i = 1, \ldots, d$ the numbers k_i and j_i denote the i-th components of the vectors $k, j \in \mathbb{Z}^d$. In this case the B_Λ are not uniformly bounded in Λ. However, the term in (26) depending on B can be estimated by putting all $b_{k,j} = 1$:

$$\sum_{k\in\Lambda^+} |b_{k,j}| \le |\Lambda_l^+| \; . \tag{30}$$

Since $|\Lambda_l^+| \le (l+g)^d \le C_\Gamma\, l^d$, where $g = \operatorname{diam}\Gamma$, we obtain the estimate

$$\mathbb{E}\left[N_\bullet^l(E_2) - N_\bullet^l(E_1)\right] \le \tilde{C}\,|E_2 - E_1|\,|\Lambda_l| \tag{31}$$

where $\tilde{C} = e^{E_2} C_V C_\Gamma\, \|f'\|_{L^1}$. The same Wegner estimate extends to similar single site potentials, e.g.

$$u = \chi_{(0,\ldots,0)} + \chi_{(1,1,0,\ldots,0)} - \chi_{(1,0,\ldots,0)} - \chi_{(0,1,0,\ldots,0)}\,. \tag{32}$$

Note that while we have proven (12) with the exponent $b = 2$ we cannot deduce the Lipschitz continuity of the IDS because of the divergent term $|\Lambda_l|^2$. This example illustrates that (H2') can be proven for A (respectively α) with inverses whose norms grow at most polynomially in $|\Lambda|$. It would be desirable to get a nice description of this class of Toeplitz matrices in terms of the convolution vector α, cf. Remark 2.

There is a completely different f then the differentiable densities considered so far we can cope with, namely the uniform density, as the following example shows.

Example 11 (Uniform density) Consider again the single site potential $u = \chi_0 - \chi_e$ but now with the uniform density $f(x) = \frac{1}{\omega_+}\chi_{[0,\omega_+]}$ for the coupling constants $\omega_k, k \in \mathbb{Z}^d$. The reasoning of Sections 3 and 4 remain valid for this case, too. Section 5 has to be replaced by explicit estimates on the volume of the integration domain M and the common density k of the transformed variables using $|\operatorname{supp} f|\,\|f\|_\infty = 1$ to get

$$\mathbb{E}\left[N_\bullet^l(E_2) - N_\bullet^l(E_1)\right] \le \text{ const } |E_2 - E_1|\,|\Lambda_l|\,. \tag{33}$$

See [46] for the details and [30] for extensions. Unfortunately we cannot deal with the superposition of the uniform and W_1^1-densities due to the transformation A^{-1} which appears in the common density k.

7 Localization

An important application of Wegner's estimate is the proof of localization. This raises the question whether there is a class of single site potentials u for which Theorem 1 is valid and additionally an initial scale estimate (H1) can be proven. As mentioned before, for non-negative u, Lifshitz tails can be used to deduce (H1) near the infimum of the spectrum of H_ω. Now, for u

with changing sign there are only restrictive results on Lifshitz asymptotics (cf. Section 6.2.2 in [20]). Most proofs are not stable under a (even small) negative perturbation of u. However, while the standard deduction of the asymptotic behaviour is based on a sequence of inequalities for the first Neumann eigenvalue of H^l_ω on arbitrarily large cubes Λ_l, (H1) is implied by this inequality on a sufficiently large, but *fixed* Λ_{Q_1}.

We will show that the basic estimate on the first Neumann eigenvalue on a fixed scale Λ_{Q_1} is stable under a negative perturbation of u as long as it is coupled with a small parameter ϵ_u. The dependence (see also Remark 17) $\epsilon_u = \epsilon_u(Q_1) \to 0$ for $Q_1 \to \infty$ explains why our estimate is no good for proving Lifshitz tails.

Throughout this section we assume that the support of f is an bounded interval. By changing the periodic potential we can assume $\operatorname{supp} f = [0, \omega_+]$. Again, details of the proofs can be found in [46].

Notation 12 (Small negative perturbation of u) We decompose $u = u_+ - \epsilon_u u_-$ into a non-negative u_+ and a non-positive part $-\epsilon_u u_-$, with $\|u_-\|_\infty \leq 1, \epsilon_u \in [0,1]$ and $\operatorname{supp} u \subset \Lambda_g, g > 0$. We set $N = \|\sum_{k \in \mathbb{Z}^d} u_-(\cdot - k)\|_\infty$.

The following arguments are adaptations of inequality (2) and Proposition 3 in [24] to u with changing sign. The restriction of H_ω and $H_0 = -\Delta + V_0$ to Λ_l with Neumann b.c. will be denoted by $H^{l,N}_\omega$ and $H^{l,N}_0$ respectively. Assume that V_0 is symmetric under the reflection along the coordinate axes. Let ϕ be the ground state of $H^{1,N}_0$ and Φ its periodic extension to $\mathbb{R}^d$. Then for $l \in \mathbb{N}$, $|\Lambda_l|^{-1/2}\, \Phi\, \chi_{\Lambda_l}$ is the ground state of both $H^{l,N}_\omega$ and $H^{l,\mathrm{per}}_\omega$, where "per" stands for periodic b.c. So we have

$$\inf \sigma(H_0) = \lambda_1\left(H^{1,\mathrm{per}}_0\right) = \lambda_1\left(H^{l,\mathrm{per}}_0\right) = \lambda_1\left(H^{l,N}_0\right) \ . \tag{34}$$

By adding a constant we get

$$\inf \sigma(H_0) = 0 \ . \tag{35}$$

Set $m_1 = \int dx\, u(x) \Phi^2(x)$ and assume that ϵ_u is so small that $m_1 > 0$. For a given energy $E \in]0,1[$ and a parameter $\beta > 0$ choose the length scale $l := [(\beta E)^{-1/2}]$.

By Dirichlet-Neumann bracketing we know

$$\mathbb{P}\{\sigma(H^l_\omega) \cap]-\infty, E[\neq \emptyset\} \leq \mathbb{P}\{\omega |\, \lambda_1(H^{l,N}_\omega) < E\}\,, \tag{36}$$

where H_ω^l may have periodic, Dirichlet or Neumann b.c. We will derive an upper bound on $\mathbb{P}\{\omega|\lambda_1(H_\omega^{l,N}) < E\}$ which is exponentially small in $|\Lambda_l| = l^d$. The exponential bound follows from the combination of a Large Deviations result and the fact that $\lambda_1(H_\omega^{l,N})$ can attain a small value only for very rare configurations of ω.

Proposition 13 *There exist $\beta_0, Q_1 < \infty$, such that for $l \geq Q_1, \beta \geq \beta_0$ and $\epsilon_u \leq \frac{E}{8\omega_+ N}$ we have the estimate:*

$$\lambda_1\left(H_\omega^l\right) < E \Longrightarrow \#\left\{k \in \Lambda_l \,\middle|\, \omega_k < \frac{4E}{m_1}\right\} > \frac{l^d}{2}\,.$$

The proof is an adaptation of the one of [23, Proposition 3] and can be found in [46]. One has just to control the contributions from u_- and is not allowed to replace u by $u\chi_0$ as done in [23].

By Large Deviations we know

$$\mathbb{P}\left\{\#\left\{k \in \Lambda_l \,|\, \omega_k < 4E/m_1\right\} > \frac{l^d}{2}\right\} \leq e^{-c|\Lambda_l|} \tag{37}$$

where we choose E sufficiently small so that $\mathbb{E}(\omega_0) > \frac{4E}{m_1}$. $c > 0$ is a constant independent of l. Combining (36), Proposition 13 and (37) we arrive at the bound

$$\mathbb{P}\{\sigma(H_\omega^l) \cap]-\infty, E[\neq \emptyset\} \leq e^{-cl^d}\,. \tag{38}$$

The relation $l \approx E^{-1/2}$ is not appropriate for the deduction of property (H1), so we have to introduce an second length scale L. Consider he operator H_ω^L on a larger cube Λ_L which is split by Neumann surfaces into cubes with side length $l := [L^{1-\zeta/2}\beta^{-1/2} - 1]$, $\zeta \in]0,1[$. The operator on the cube $\Lambda_l + j$ for $j \in (l\mathbb{Z})^d \cap \Lambda_L$ is denoted by $H_{\omega,j}$. We have

$$\lambda_1(H_\omega^L) \geq \inf_{j \in (l\mathbb{Z})^d \cap \Lambda_L} \lambda_1(H_{\omega,j}).$$

and thus using (38)

$$\begin{aligned}\mathbb{P}\{\,\lambda_1(H_\omega^{L,N}) < L^{-2+\zeta}\} &\leq \mathbb{P}\{\,\lambda_1(H_\omega^{L,N}) \leq \beta^{-1}(l+1)^{-2}\} \\ &\leq \mathbb{P}\left\{\inf_{j\in(l\mathbb{Z})^d\cap\Lambda_L} \lambda_1(H_{\omega,j}) \leq \beta^{-1}(l+1)^{-2}\right\} \\ &\leq \sum_{j\in(l\mathbb{Z})^d\cap\Lambda_L} \mathbb{P}\{\,\lambda_1(H_{\omega,j}) \leq \beta^{-1}(l+1)^{-2}\} \\ &\leq \left(\frac{L}{l}\right)^d \mathbb{P}\{\,\lambda_1(H_{\omega,0}) \leq \beta^{-1}(l+1)^{-2}\} \\ &\leq \left(\frac{L}{l}\right)^d e^{-\left(\frac{11}{12}\right)^2\frac{l^d}{2}} \leq L^{-q} \end{aligned} \tag{39}$$

for any $\zeta \in]0,1[$ and $q \in \mathbb{N}$ for L large enough.

Note that due to the condition in Proposition 13 the last inequality is applicable for single site potentials $u = u_+ - \epsilon_u u_-$ with $\epsilon_u \leq (8\omega_+ N\beta(l+1)^2)^{-1}$. Applying the Combes-Thomas argument (cf. [7], [1], [26, Appendix] or [42, Section 2.4]) the initial scale estimate (H1) follows.

Note that by [26, equation (1.1)] $\sigma(H_\omega) \supset \sigma(H_0) \ni 0$. This implies that for any $E > 0$ the interval $[0, E]$ actually does contain spectrum. Now localization at the bottom of the spectrum follows.

Theorem 14 *Let H_ω be as in Theorem 1. Assume that V_0 is symmetric under the reflection along the coordinate axes and* $\operatorname{supp} f = [0, \omega_+]$. *Then there exist $\epsilon_u > 0$ and $E^* > 0 = \inf\sigma(H_0)$ such that sufficiently small ϵ_u:*

$$\sigma(H_\omega) \cap]-\infty, E^*[\neq \emptyset, \;\; \sigma_c(H_\omega) \cap]-\infty, E^*[= \emptyset. \tag{40}$$

We turn our attention now to localization away from the infimum of $\sigma(H_\omega)$. The spectrum of the periodic Schrödinger operator H_0 consists of intervals called spectral bands. If there are gaps belonging to the resolvent set between them, there exist internal spectral (band) edges. If the perturbation V_ω is small, the spectral gaps will be preserved, although with shifted spectral edges. It is natural to ask whether the localization results for energies near $\inf\sigma(H_\omega)$ can be extended to small neighbourhoods of internal edges. It turns out that the study of Lifshitz tails in this energy regime is quite involved [29].

As a substitute for the Lifshitz asymptotic a special disorder regime has been assumed in several papers [1, 26]. In this case the density f of the

coupling constants is required to satisfy

$$\int_{\omega_-}^{\omega_-+\delta} f(x)\,dx \le \delta^\tau \text{ or } \int_{\omega_+-\delta}^{\omega_+} f(x) \le \delta^\tau \text{ for some } \tau > d/2 \text{ and small } \delta. \tag{41}$$

The first condition is needed when considering lower band edges, the second for upper band edges. In this case we can prove using the arguments of [26, pp.10-11]:

Proposition 15 *Let H_ω, f be as in Theorem 14, let f satisfy (41) and E be a spectral band edge. Let $p \in]0, 2\tau - d[$ and $\xi \in]0, 2 - \frac{d+p}{\tau}[$. Then there exists a Q_1 such that for all $l \ge Q_1$ and $\epsilon_u \le \frac{l^{\xi-2}}{\omega_+ N}$ we have*

$$\mathbb{P}\{\sigma(H_\omega^{l,per}) \cap [E - l^{\xi-2}, E + l^{\xi-2}] \neq \emptyset\} \le l^{-p}\,. \tag{42}$$

Now one proceeds as in [1, 26] or [42] using Combes-Thomas arguments to prove an initial scale estimate and thereby localization.

Theorem 16 *Let H_ω and f be as in Proposition 15 and let be E a spectral band edge. There exist $r > 0$ such that for sufficiently small ϵ_u the spectrum of H_ω in the interval $[E - r, E + r]$ is pure point.*

In Section 6.2 of [20] it is described how to prove this result for a larger class of single site potentials and density functions f, using results from [29] and an abstract version of the smallness of u_-. To apply this reasoning it is necessary that the unperturbed periodic operator $-\Delta + V_0$ is non-degenerate (resp. Floquet-regular) at the considered spectral boundary.

Remark 17 For the localization Theorems 14 and 16 we had to choose ϵ_u small depending on the initial scale Q_1. It might seem irritating that Q_1 in turn depends on V_ω, i.e. implicitly on ϵ_u. However an admissible initial scale Q_1 for $\epsilon_u = 1$, i.e. the potential $V_\omega^{\epsilon_u=1}(x) = \sum_{k\in\mathbb{Z}^d} \omega_k\,(u_+ - u_-)(x-k)$ is admissible for the potentials for all values of $\epsilon_u \in [0, 1]$, also. This means that choosing ϵ_u closer to 0 does not change the initial scale Q_1.

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